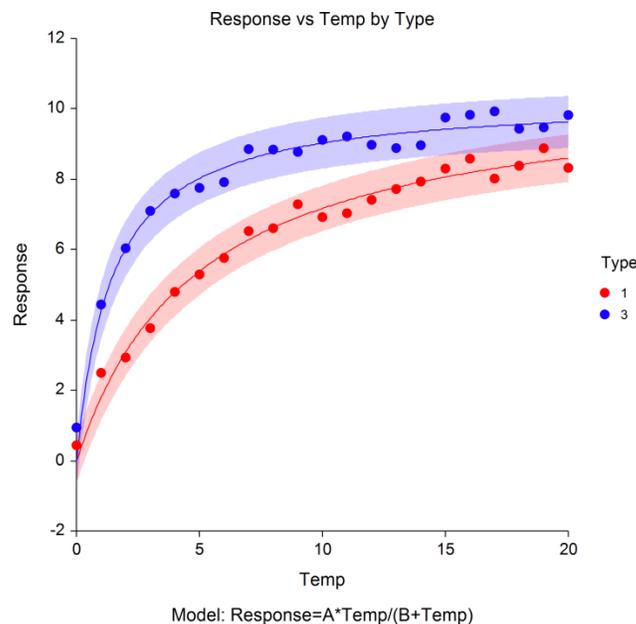


## Chapter 351

# Curve Fitting – General

## Introduction

*Curve fitting* refers to finding an appropriate mathematical model that expresses the relationship between a dependent variable  $Y$  and a single independent variable  $X$  and estimating the values of its parameters using nonlinear regression. An introduction to curve fitting and nonlinear regression can be found in the chapter entitled Curve Fitting, so these details will not be repeated here. Here are some examples of the curve fitting that can be accomplished with this procedure.



This program is general purpose curve fitting procedure providing many new technologies that have not been easily available. It is preprogrammed to fit over forty common mathematical models including growth models like linear-growth and Michaelis-Menten. It also fits many approximating models such as regular polynomials, piecewise polynomials, and polynomial ratios. In addition to these preprogrammed models, it also fits models that you write yourself.

This routine includes several innovative features. First, it can fit curves to several batches of data simultaneously. Second, it compares fitted models across groups using graphics and numerical tests such as an approximate F-test for curve coincidence and a computer-intensive randomization test that compares curve coincidence and individual parameter values. Third, this procedure computes bootstrap confidence intervals for parameter values, predicted means, and predicted values using the latest computer-intensive bootstrapping technology.

## Preset Models

Below are the models that you may fit. Select 'Custom' to use a model you have entered in the 'Custom Model' box. Whenever possible, use one of the preset models since reasonable starting values for the parameters will be calculated for you. The minimum, maximum, and starting values of each letter in the preset model are defined in the corresponding MIN START MAX box on the Options panel. The preset models available are

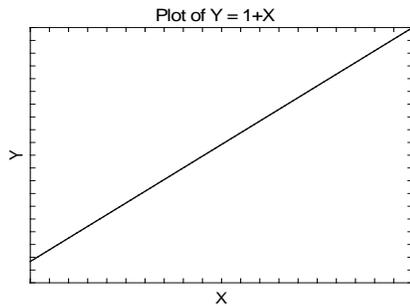
0	Custom	Use the custom model
1	$Y=A+BX$	Simple Linear
2	$Y=A+BX+CX^2$	Quadratic
3	$Y=A+BX+CX^2+DX^3$	Cubic
4	$Y=(A+BX)/(1+CX)$	PolyRatio(1,1)
5	$Y=(A+BX+CX^2)/(1+DX+EX^2)$	PolyRatio(2,2)
6	$Y=(A+BX+CX^2+DX^3)/(1+EX+FX^2+GX^3)$	PolyRatio(3,3)
7	$Y=(A+BX+CX^2+DX^3+EX^4) / (1+FX+GX^2+HX^3+IX^4)$	PolyRatio(4,4)
8	$Y=AX/(B+X)$	Michaelis-Menten
9	$Y=1/(A+BX)$	Reciprocal
10	$Y=(A+BX)^{-1/C}$	Bleasdale-Nelder
11	$Y=1/(A+BX^C)$	Farazdaghi and Harris
12	$Y=1/(A+BX+CX^2)$	Holliday
13	$Y=EXP(A(X-B))$	Exponential
14	$Y=A(1-EXP(-B(X-C)))$	Monomolecular
15	$Y=A/(1+B(EXP(-CX)))$	Three Parameter Logistic
16	$Y=D+(A-D)/(1+B(EXP(-CX)))$	Four Parameter Logistic
17	$Y=A(EXP(-EXP(-B(X-C))))$	Gompertz
18	$Y=A-(A-B)EXP(-(C X )^D)$	Weibull
19	$Y=A-(A-B)/(1+(C X )^D)$	Morgan-Mercer-Floding
20	$Y=A(1+(B-1)EXP(-C(X-D)))^{1/(1-B)}$	Richards
21	$Y=B(LN( X -A))$	Logarithmic
22	$Y=A(1-B^X)$	Power
23	$Y=AX^{BX^C}$	Power^Power
24	$Y=A(EXP(-BX))+C(EXP(-DX))$	Sum of Exponentials
25	$Y=A(X^B)EXP(-CX)$	Exponential Type 1
26	$Y=(A+BX)EXP(-CX)+D$	Exponential Type 2
27	$Y=A+B(EXP(-C(X-D)^2))$	Normal
28	$Y=A+(B/X)EXP(-C(LN( X )-D)^2)$	Lognormal
29	$Y=A \text{ Exp}(-BX)$	Exponential
30	$Y=AX/(B+X) + CX/(D+X)$	Michaelis-Menten(2)
31	$Y=AX/(B+X) + CX/(D+X) + EX/(F+X)$	Michaelis-Menten(3)
32	$Y=A + BX + C(X-D)SIGN(X-D)$	Linear-Linear
33	$Y=A+BX+CX^2+(X-D)SIGN(X-D)[C(X+D)+E]$	Linear-Quadratic
34	$Y=A+BX+CX^2+(X-D)SIGN(X-D)[E(X+D)+F]$	Quadratic-Linear
35	$Y=A+BX+CX^2+(X-D)SIGN(X-D)[E(X+D)+F]$	Quadratic-Quadratic
36	$Y=A+BX+C(X-D)SIGN(X-D)+E(X-F)SIGN(X-F)$	Linear-Linear-Linear
37	$Y=Exp((A/B)(1-Exp(BX)))$	Gompertz 2
38	$Y=AX^C/(B^C+X^C)$	Hill

## Selecting a Preset Model

Over thirty preset models are available. These models provide a variety of curve shapes. Several of the models were developed for quite different physical processes but yield similar results. We now present examples and details of several of the preset models available.

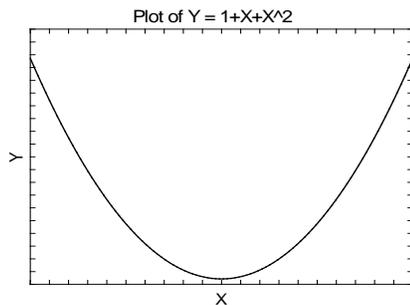
### 1. Linear: $Y=A+BX$

This common model is usually fit using standard linear regression techniques. We include it here to allow for various special forms made by transforming  $X$  and  $Y$



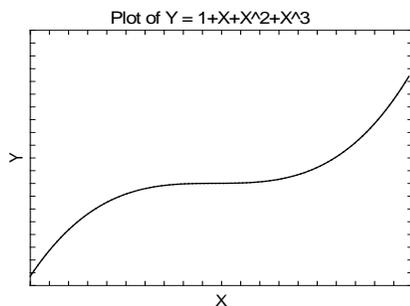
### 2. Quadratic: $Y=A+BX+CX^2$

The quadratic or second-order polynomial model results in the familiar parabola.



### 3. Cubic: $Y=A+BX+CX^2+DX^3$

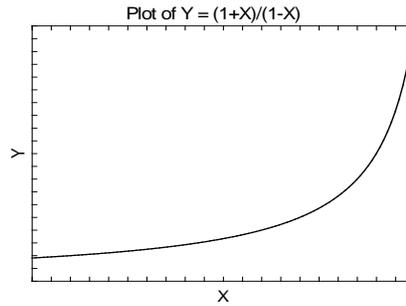
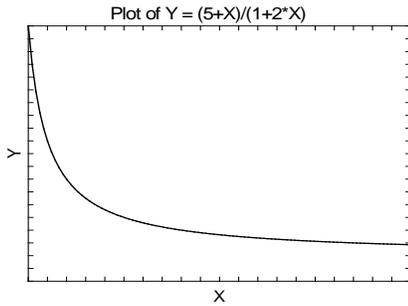
This is the cubic or third-order polynomial model.



Curve Fitting – General

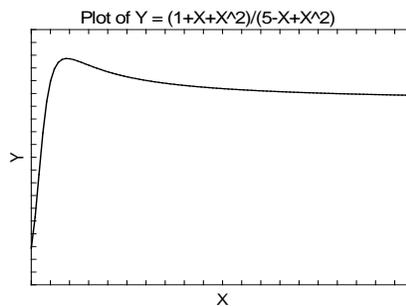
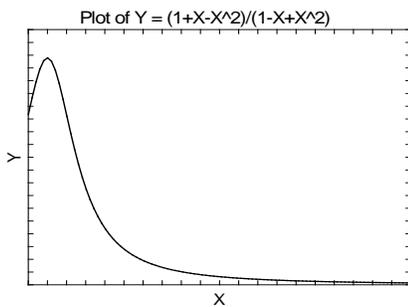
**4. PolyRatio(1,1):  $Y=(A+BX)/(1+CX)$**

The ratio of first-order polynomials model is a slight extension of the Michaelis-Menten model. It may be used to approximate many more complicated models.



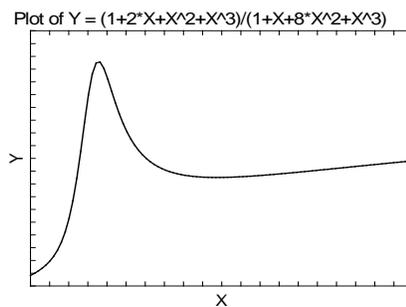
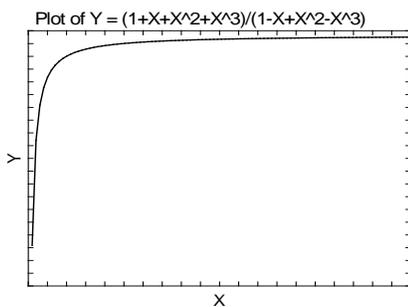
**5. PolyRatio(2,2):  $Y=(A+BX+CX^2)/(1+DX+EX^2)$**

The ratio of second-order polynomials model may be used to approximate many complicated models.



**6. PolyRatio(3,3):  $Y=(A+BX+CX^2+DX^3)/(1+EX+FX^2+GX^3)$**

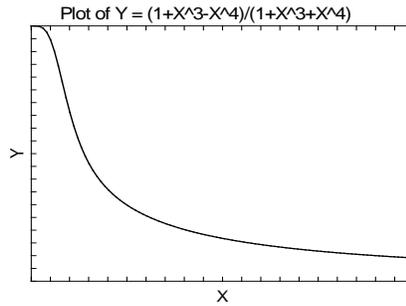
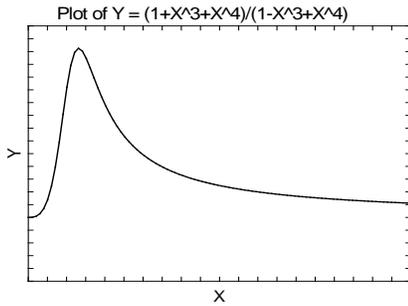
The ratio of third-order polynomials model may be used to approximate many complicated models. However, care must be used when estimating such high-degree models.



Curve Fitting – General

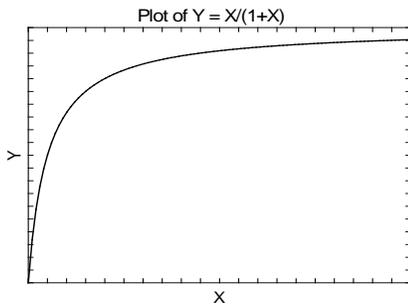
**7. PolyRatio(4,4):  $Y=(A+BX+CX^2+DX^3+EX^4) / (1+FX+GX^2+HX^3+IX^4)$**

The ratio of fourth-order polynomials model may be used to approximate many complicated models. However, care must be used when estimating such high-degree models.



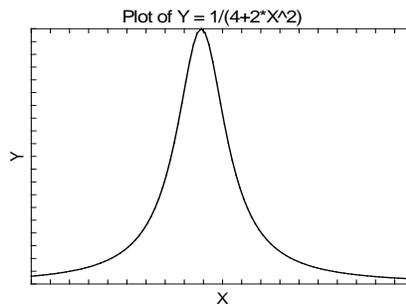
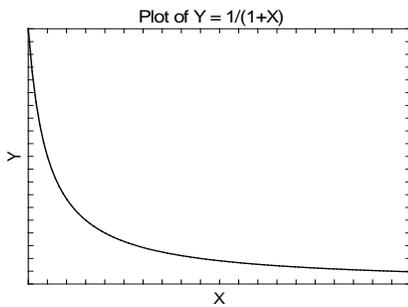
**8. Michaelis-Menten:  $Y=AX/(B+X)$**

This is a popular growth model.



**9. Reciprocal:  $Y=1/(A+BX)$**

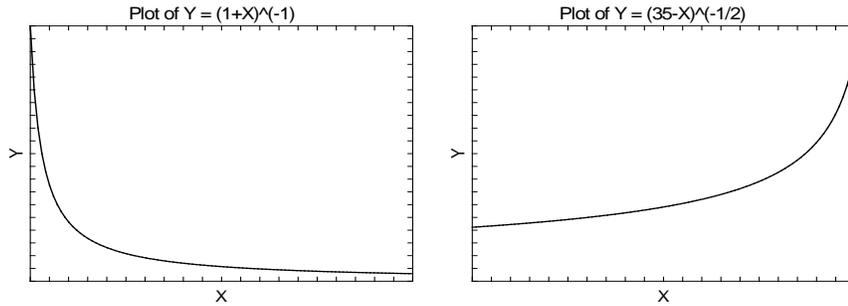
This model, known as the reciprocal or Shinozaki and Kira model, is mentioned in Ratkowsky (1989, page 89) and Seber (1989, page 362).



Curve Fitting – General

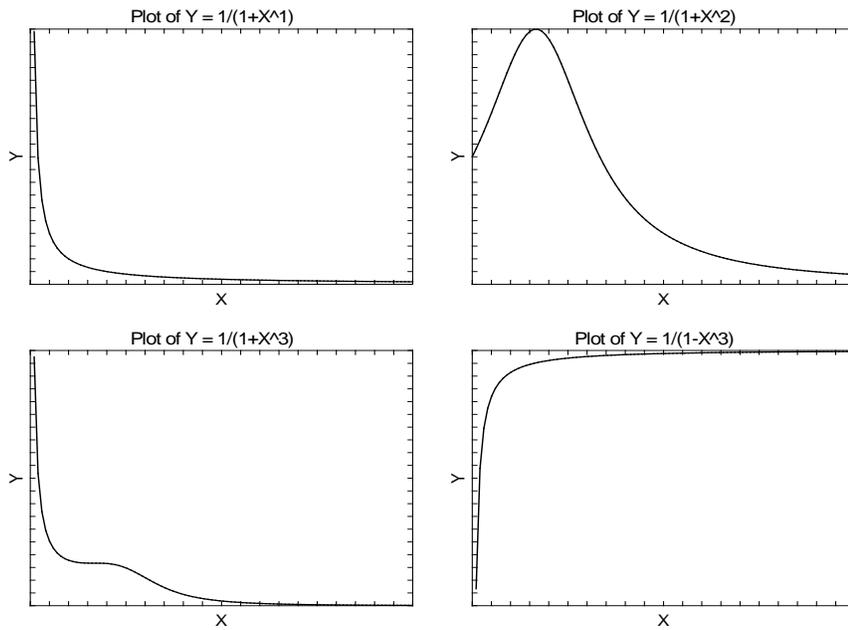
**10. Bleasdale-Nelder:  $Y=(A+BX)^{-1/C}$**

This model, known as the Bleasdale-Nelder model, is mentioned in Ratkowsky (1989, page 103) and Seber (1989, page 362).



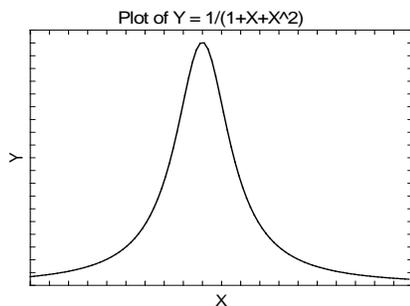
**11. Farazdaghi and Harris:  $Y=1/(A+BX^C)$**

This model, known as the Farazdaghi and Harris model, is mentioned in Ratkowsky (1989, pages 99 and 104) and Seber (1989, page 362).



**12. Holliday:  $Y=1/(A+BX+CX^2)$**

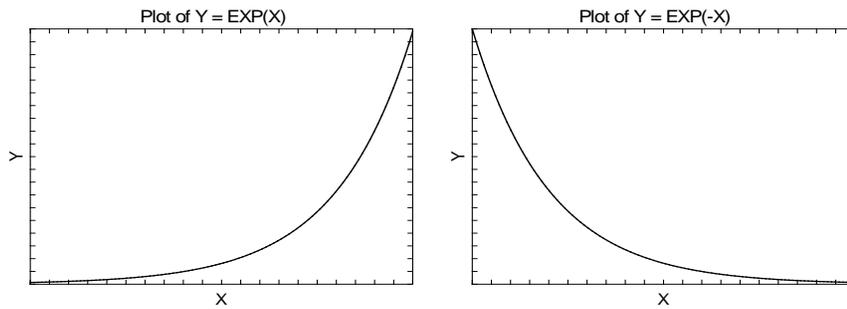
This model, known as the Holliday model, is mentioned in Seber (1989, page 362).



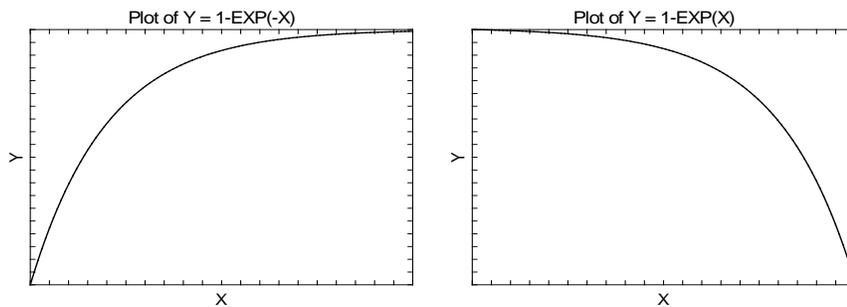
## Curve Fitting – General

**13. Exponential:  $Y=EXP(A(X-B))$** 

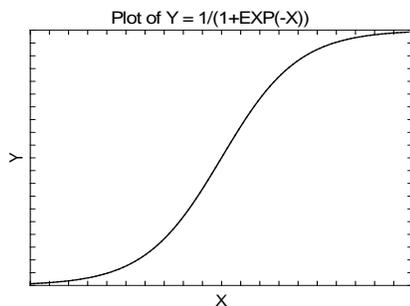
This model, known as the exponential model, is mentioned in Seber (1989, page 327). Note that taking the log of both sides reduces this equation to a linear model.

**14. Monomolecular:  $Y=A(1-EXP(-B(X-C)))$** 

This model, known as the monomolecular model, is mentioned in Seber (1989, page 328).

**15. Three Parameter Logistic:  $Y=A/(1+B(EXP(-CX)))$** 

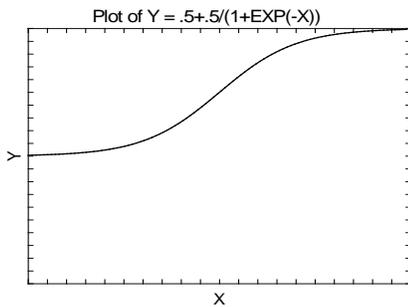
This model, known as the three-parameter logistic model, is mentioned in Seber (1989, page 330).



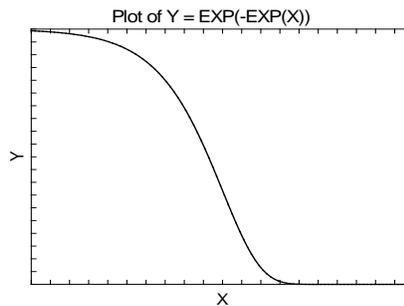
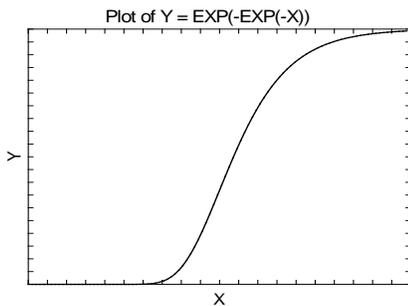
## Curve Fitting – General

**16. Four Parameter Logistic:  $Y=D+(A-D)/(1+B(\text{EXP}(-CX)))$** 

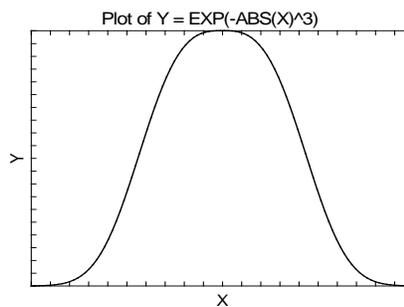
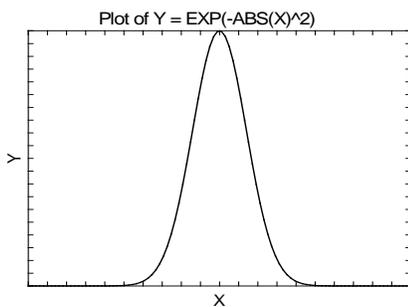
This model, known as the four-parameter logistic model, is mentioned in Seber (1989, page 338). Note that the extra parameter, D, has the effect of shifting the graph vertically. Otherwise, this plot is the same as the three-parameter logistic.

**17. Gompertz:  $Y=A(\text{EXP}(-\text{EXP}(-B(X-C))))$** 

This model, known as the Gompertz model, is mentioned in Seber (1989, page 331).

**18. Weibull:  $Y=A-(A-B)\text{EXP}(-(C|X|)^D)$** 

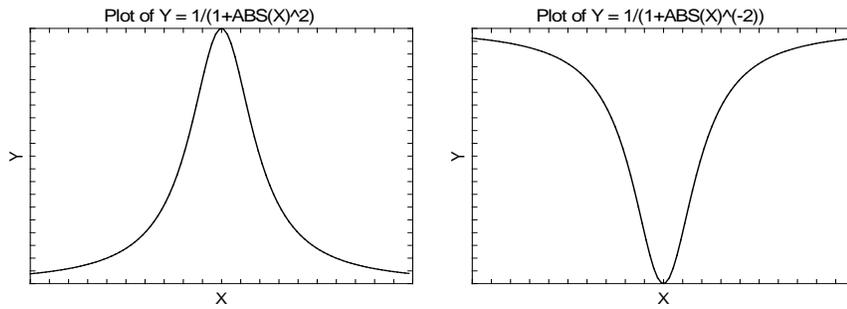
This model, known as the Weibull model, is mentioned in Seber (1989, page 338).



Curve Fitting – General

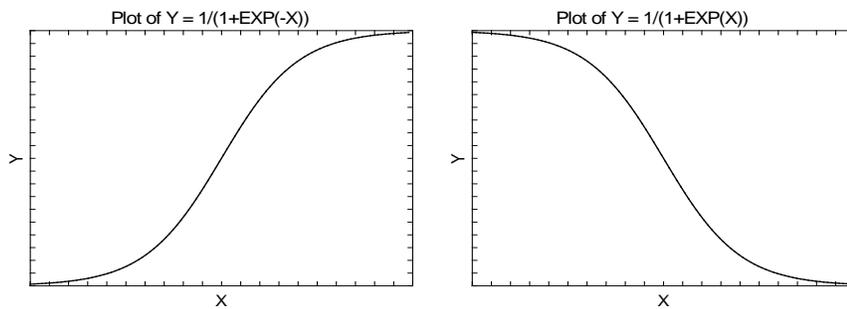
**19. Morgan-Mercer-Floding:  $Y=A-(A-B)/(1+(C|X|)^D)$**

This model, known as the Morgan-Mercer-Floding model, is mentioned in Seber (1989, page 340).

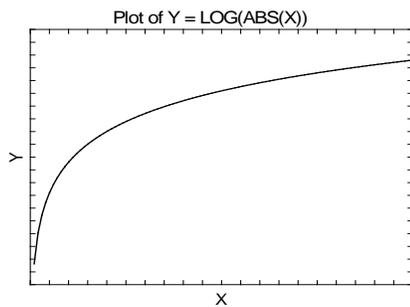


**20. Richards:  $Y=A(1+(B-1)EXP(-C(X-D)))^{1/(1-B)}$**

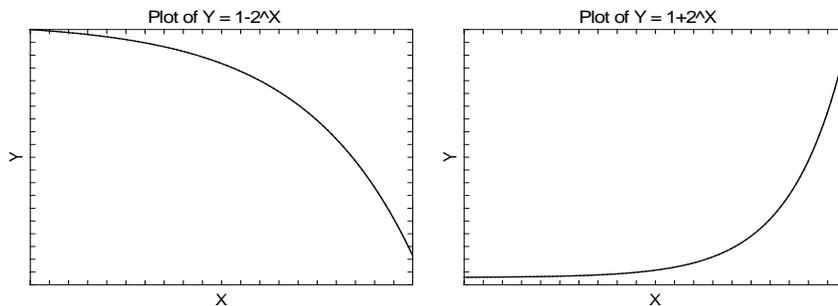
This model, known as the Richards model, is mentioned in Seber (1989, page 333).



**21. Logarithmic:  $Y=B(LN(|X|-A))$**

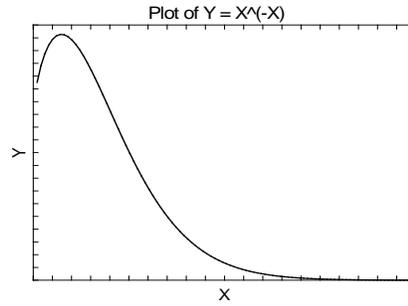
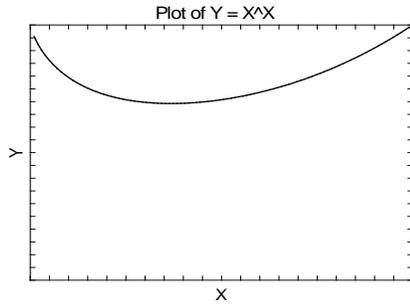


**22. Power:  $Y=A(1-B^X)$**

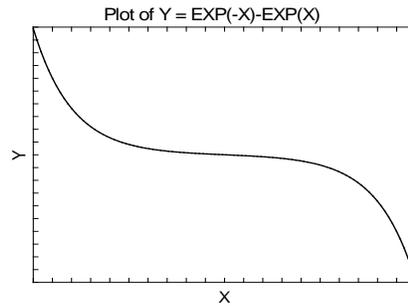
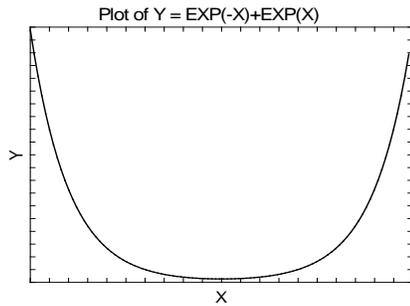


Curve Fitting – General

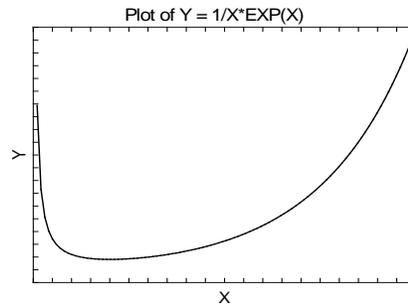
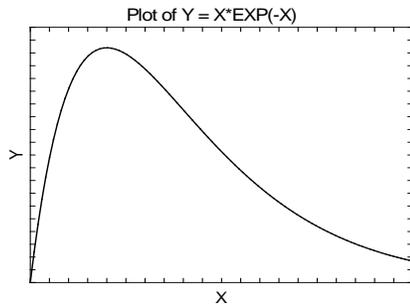
**23. Power^Power:  $Y=AX^{(BX^C)}$**



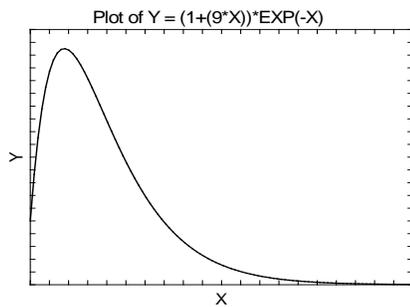
**24. Sum of Exponentials:  $Y=A(\text{EXP}(-BX))+C(\text{EXP}(-DX))$**

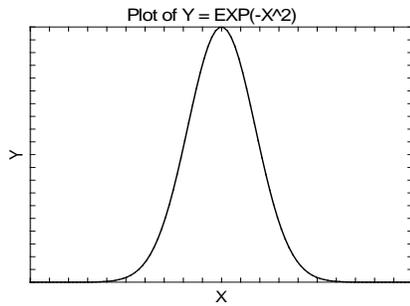
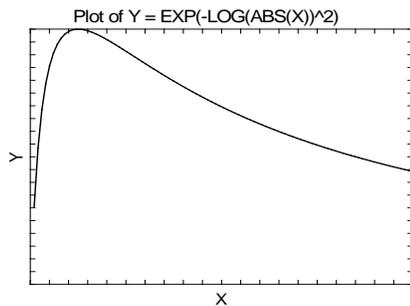
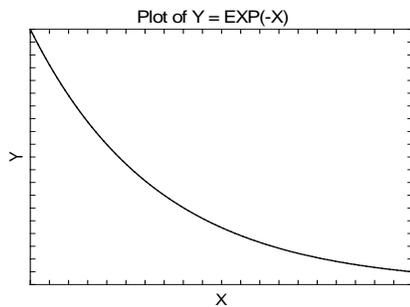
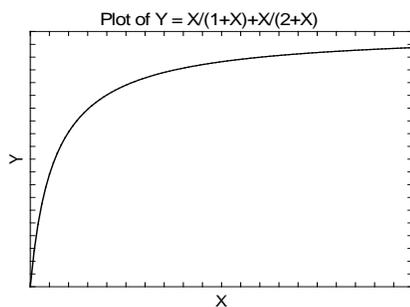


**25. Exponential Type 1:  $Y=A(X^B)\text{EXP}(-CX)$**



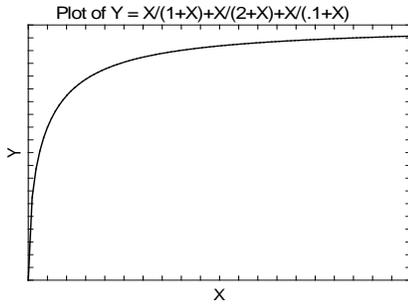
**26. Exponential Type 2:  $Y=(A+BX)\text{EXP}(-CX)+D$**



**27. Normal:  $Y=A+B(\text{EXP}(-C(X-D)^2))$** **28. Lognormal:  $Y=A+(B/X)\text{EXP}(-C(\text{LN}(|X|)-D)^2)$** **29. Exponential:  $Y=A \text{Exp}(-BX)$** **30. Michaelis-Menten(2):  $Y=AX/(B+X) + CX/(D+X)$** 

Curve Fitting – General

**31. Michaelis-Menten(3):  $Y=AX/(B+X) + CX/(D+X) + EX/(F+X)$**



**32. Linear-Linear:  $Y=A + BX + C(X-D)SIGN(X-D)$**

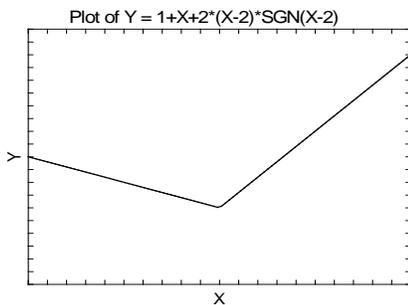
**Common Equation**

$Y = a1 + b1X, \quad X \leq J$

$Y = a2 + b2X, \quad X > J$

**Parameter Identities**

$A = (a1 + a2)/2$	$B = (b1 + b2)/2$	$a1 = A + DC$	$b1 = B - C$	$J = D$
$C = (b2 - b1)/2$	$D = J$	$a2 = A - DC$	$b2 = B + C$	



**33. Linear-Quadratic:  $Y=A+BX+CX^2+(X-D)SIGN(X-D)[C(X+D)+E]$**

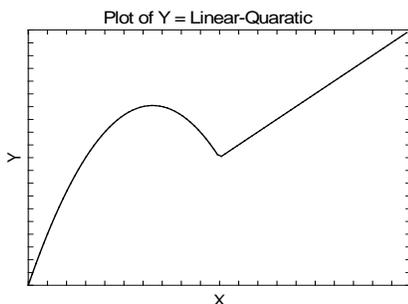
**Common Equation**

$Y = a1 + b1X, \quad X \leq a$

$Y = a2 + b2X + c2X^2, \quad X > a$

**Parameter Identities**

$A = (a1 + a2)/2$	$B = (b1 + b2)/2$	$C = c2/2$
$D = a$	$E = (b2 - b1)/2$	
$a1 = A + CD^2 + DE$	$b1 = B - E$	$a = D$
$a2 = A - CD^2 - DE$	$b2 = B + E$	$c2 = 2C$



Curve Fitting – General

**34. Quadratic-Linear:  $Y=A+BX+CX^2+(X-D)SIGN(X-D)[E(X+D)+F]$**

**Common Equation**

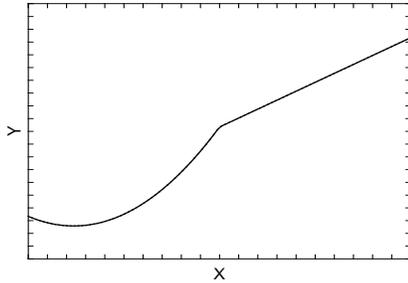
$Y = a_1 + b_1X + c_1X^2, \quad X \leq a$

$Y = a_2 + b_2X, \quad X > a$

**Parameter Identities**

$A = (a_1 + a_2)/2$	$B = (b_1 + b_2)/2$	$C = c_1/2$
$D = a$		$E = (b_2 - b_1)/2$
$a_1 = A - CD^2 + DE$	$b_1 = B - E$	$a = D$
$a_2 = A + CD^2 - DE$	$b_2 = B + E$	$c_1 = 2C$

Plot of Y = Linear-Quadratic



**35. Quadratic-Quadratic:  $Y=A+BX+CX^2+(X-D)SIGN(X-D)[E(X+D)+F]$**

**Common Equation**

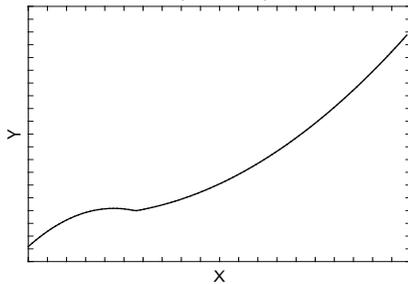
$Y = a_1 + b_1X + c_1X^2, \quad X \leq a$

$Y = a_2 + b_2X + c_2X^2, \quad X > a$

**Parameter Identities**

$A = (a_1 + a_2)/2$	$B = (b_1 + b_2)/2$	$C = (c_1 + c_2)/2$
$D = a$	$E = (c_2 - c_1)/2$	$F = (b_2 - b_1)/2$
$a_1 = A - ED^2 + DF$	$b_1 = B - F$	$a = D$
$a_2 = A + ED^2 - DF$	$b_2 = B + F$	
$c_1 = C - E$	$c_2 = C + E$	

Plot of Y = Quadratic-Quadratic



Curve Fitting – General

**36. Linear-Linear-Linear:  $Y=A+BX+C(X-D)SIGN(X-D)+E(X-F)SIGN(X-F)$**

**Common Equation**

$Y = a_1 + b_1X, \quad X \leq J_1$

$Y = a_2 + b_2X, \quad J_1 < X \leq J_2$

$Y = a_3 + b_3X, \quad X > J_2$

**Parameter Identities**

$A = (a_1 + a_3)/2 \quad B = (b_1 + b_3)/2 \quad C = (b_2 - b_1)/2$

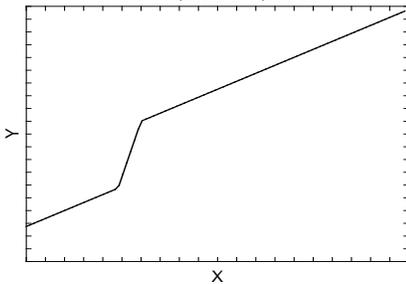
$D = J_1 \quad E = (b_3 - b_2)/2 \quad F = J_2$

$a_1 = A + CD + EF \quad b_1 = B - C - E \quad J_1 = D$

$a_2 = A - CD - EF \quad b_2 = B + C - E \quad J_2 = F$

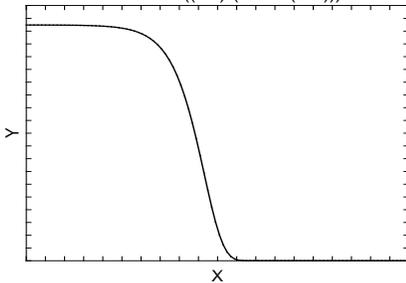
$a_3 = A - CD + EF \quad b_3 = B + C + E$

Plot of Y = Quadratic-Quadratic



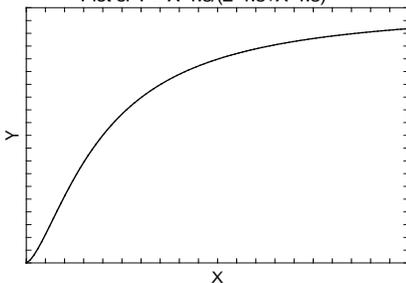
**37. Gompertz 2:  $Y=Exp((A/B)(1-Exp(BX)))$**

Plot of Y = EXP((4/2)\*(1-EXP(2\*X)))



**38. Hill:  $Y=AX^C/(B^C+X^C)$**

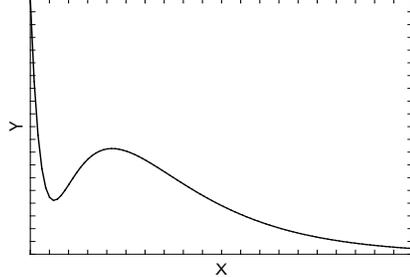
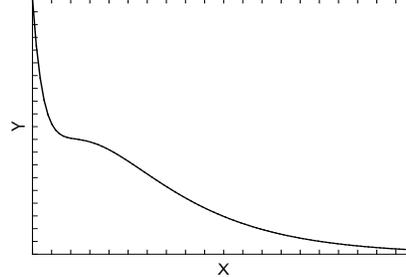
Plot of Y = X^1.5/(2^1.5+X^1.5)



## Curve Fitting – General

**39. Sum of 3 Exponentials:  $Y=A(\text{Exp}(-BX))-C(\text{Exp}(-DX))+E(\text{Exp}(-FX))$** 

This model is intended for the case when all parameters are positive. Note that the default starting values may not work for this model. You should be prepared to try different starting values.

Plot of  $Y = 2 \cdot \text{Exp}(-.9 \cdot X) - 3 \cdot \text{Exp}(-2 \cdot X) + 2 \cdot \text{Exp}(-6 \cdot X)$ Plot of  $Y = 3 \cdot \text{Exp}(-.9 \cdot X) - 3 \cdot \text{Exp}(-2 \cdot X) + 2 \cdot \text{Exp}(-6 \cdot X)$ 


---

## Custom Models

You are not limited to the preset models that are shown above. You can enter your own custom model using standard mathematical notation. The only difference between using a preset model and using your own model is that with a preset model the starting values of the search algorithm are chosen based on the model. When using a custom model, you will have to set your own starting values based on the data you are trying to fit. When you do not specify starting values, the program uses all zeros, which may or may not lead to a reasonable solution.

---

## Using a Custom Model

The custom model is only used when the Preset Model option is set to 'Custom Model'. When used, it contains the regression model written in standard mathematical notation.

Use 'X' to represent the independent variable specified in the X Variable box, not its variable name. Hence, if your independent variable is HEAT, you would enter  $A+B \cdot \text{LN}(X)$ , not  $A+B \cdot \text{LN}(\text{HEAT})$ .

Use the letters (case ignored) A,B,C, ... (except X and Y) to represent the parameters to be estimated from the data. The letters used must be specified in one of the Parameter boxes listed under the Search tab. Note that you do not include a 'Y=' in the expression. That is, you would enter  $A+B \cdot X$ , not  $Y=A+B \cdot X$ .

## Expression Syntax

Construct the expression using standard mathematical syntax. Possible symbols and functions are

### Symbols

+	add
-	subtract
*	multiply
/	divide
^	exponent ( $X^2 = X \cdot X$ )
()	parentheses
<	less than.
>	greater than
=	equals
<=	less than or equal
>=	greater than or equal
<>	not equal

## Curve Fitting – General

## Functions

(a logic b)	Indicator function. If true, result is 1; otherwise, result is 0. Logic values are <, >, =, <>, <=, and >=.
ABS(X)	Absolute value of X.
ARCOSH(X)	Arc cosh of X.
ARSINH(X)	Arc sinh of X.
ARTANH(X)	Arc tanh of X.
ASN(X)	Arc sine of X.
ATN (X)	Arc tangent of X.
COS(X)	Cosine of X.
COSH(X)	Hyperbolic cosine of X.
ERF(X)	The error function of X
EXP(X)	Exponential of X.
INT(X)	Integer part of X.
LN(X)	Log base e of X.
LOG(X)	Log base 10 of X.
LOGGAMMA(X)	Log of the gamma function.
NORMDENS(X)	Normal density.
NORMPROB(X)	Normal CDF (probability).
NORMVALUE(X)	Inverse normal CDF.
SGN(X)	Sign of X which is -1 if X<0, 0 if X=0, and 1 if X>0.
SIN(X)	Sine of X.
SINH(X)	Hyperbolic sine of X.
SQR(X)	Square root of X.
TAN(X)	Tangent of X.
TANH(X)	Hyperbolic tangent of X.
TNH(X)	Hyperbolic tangent of X.
TRIGAMMA(X)	Trigamma function.

## Independent Variable

Use 'X' in your expression to represent the independent variable you have specified.

## Parameters

The letters of the alphabet (except X and Y) may be used to represent the parameters. Parameters can be only one character long and case is ignored. Each parameter must be defined in the Parameter fields below.

## Numbers

You can enter numbers in standard format such as 23.456 and 254.43, or you can use scientific notation such as 1E-5 (which is 0.00001) and 1E5 (which is 100000).

## Examples

Standard mathematical syntax is used. This is discussed in detail in the Transformation section. Examples of valid expressions are:

$A+B*X$

$C+D*X+E*X*X$  or  $G+H*X+B*X^2$

$A*EXP(B*X)$

$(X<=5)*A+(X>5)*B+C$

## Confidence Intervals

Two methods are used to calculate confidence intervals of the regression parameters and predicted values. The first method is based on the usual normality and constant variance of residuals assumption. When the data follow these assumptions, standard expressions for the confidence intervals are used based on the Student's  $t$  distribution. Unfortunately, nonlinear regression datasets rarely follow these assumptions.

The second method is called the *bootstrap* method. This is a modern, computer-intensive method that has only become available in recent years as extensive computer power has become available.

## Bootstrap Confidence Intervals

*Bootstrapping* provides standard errors and confidence intervals for nonlinear-regression parameter, predicted means, and predicted values. The method is simple in concept, but it requires extensive computation time.

Bootstrap confidence intervals are based on the assumption that your sample is actually representative of the population. Beginning with this assumption,  $B$  samples are drawn ( $B$  is over 1000) of size  $N$  from your original sample with replacement. With replacement sampling means that each observation may be selected more than once. For each bootstrap sample, the nonlinear-regression results are computed and stored.

Suppose you want the standard error and a confidence interval of a regression parameter. The bootstrap sampling process provides  $B$  estimates of this parameter. The standard deviation of these  $B$  estimates is the bootstrap estimate of the standard error of the parameter. The bootstrap confidence interval is found by arranging the  $B$  values in sorted order and selecting the appropriate percentiles from the list. For example, a 90% bootstrap confidence interval for the parameter is given by fifth and ninety-fifth percentiles of the bootstrap parameter values.

The main assumption made when using the bootstrap is that your sample approximates the population. Because of this assumption, bootstrapping does not work well for small samples in which there is little likelihood that the sample is representative of the population. Bootstrapping should only be used in medium to large samples.

## Bootstrap Prediction Intervals

Bootstrap confidence intervals for the mean of  $Y$  given  $X$  are generated from the bootstrap sample in the usual way. To calculate prediction intervals for the predicted value (not the mean) of  $Y$  given  $X$  requires a modification to the predicted value of  $Y$  to be made to account for the variation of  $Y$  about its mean. This modification of the predicted  $Y$  values in the bootstrap sample, suggested by Davison and Hinkley, is as follows.

$$\hat{y}_i = \hat{y}_i + e_r^*$$

where  $e_r^*$  is a randomly selected modified residual (see below). By adding the residual we have added an appropriate amount of variation to represent the variance of individual  $Y$ 's about their mean value.

## Modified Residuals

Davison and Hinkley (1999) page 279 recommend the use of a special rescaling of the residuals when bootstrapping to keep results unbiased. Because of the high amount of computing involved in bootstrapping, these modified residuals are calculated using

$$e_j^* = \frac{e_j}{\sqrt{1 - \frac{1}{N}}} - \bar{e}$$

## Curve Fitting – General

where

$$\bar{e} = \frac{\sum_{j=1}^N e_j}{N}$$

Note that there is a different rescaling than Davison and Hinkley recommended. We have used this rescaling because it is much quicker to calculate.

---

## Hypothesis Testing

When curves are fit to two or more groups, it is often of interest to test whether certain regression parameters are equal and whether the fitted curves coincide. Although some approximate results have been obtained using indicator variables, these are asymptotic results, and little is known about their appropriateness in sample samples. We provide a test of the hypothesis that all group curves coincide using an  $F$ -test that compares the residual sum of squares obtained when the grouping is

ignored with the total of the residual sum of squares obtained for each group. This test is routinely used in the analysis of variance associated with linear models and its application to nonlinear models has occasionally been suggested. However, it is based on naive assumptions that seldom occur.

Because of the availability of fast computing speed in recent years, a second method of hypothesis testing, called the *randomization test*, is now available. This test will be discussed next.

---

## Randomization Test

Randomization testing is discussed by Edgington (1987). The details of the randomization test are simple: all possible permutations of the group variable while leaving the dependent and independent variables in their original order are investigated. For each permutation, the difference between the estimated group parameters is calculated. The number of permutations with a magnitude greater than or equal to that of the actual sample is counted. Dividing this count by the number of permutations gives the significance level of the test.

The randomization test is suggested because an exact test is achieved without making unrealistic assumptions about the data such as constant variance, normality, or model accuracy. The test was not used in the past because the number of computations was prohibitive. In fact, the randomization test was originally proposed by Fisher and he chose his  $F$ -test because its distribution close approximated the randomization distribution.

The only assumption that a randomization test makes is that the data values are *exchangeable* under the null hypothesis.

For even moderate sample sizes, the total number of permutations is in the trillions, so a Monte Carlo approach is used in which the permutations are found by random selection rather than enumeration. Using this approach, a reasonable approximation to the test's probability level may be found by considering only a few thousand permutations rather than the trillions needed for complete enumeration. Edgington suggests that at least 1000 permutations be computed. We suggest that this be increased to 10000 for important results.

The program tests two types of hypotheses using randomization tests. The first is that each of the estimated model parameters is equal. The second is that the individual fitted curves coincide across all groups.

## Curve Fitting – General

### Randomization Statistics for Testing Parameter Equivalence

The test statistic for comparing a model parameter is formed by summing the difference between the group parameter estimates for each pair of groups. If there are  $G$  groups, the test statistic is computed using the formula

$$B_{RT} = \sum_{i=1}^{G-1} \sum_{j=i+1}^G \left| \hat{\beta}_i - \hat{\beta}_j \right|$$

### Randomization Statistics for Testing Curve Equivalence

The test statistic for comparing the whole curve is formed by summing the difference between the estimated predicted values for each pair of groups at several points along the curve. If there are  $G$  groups and  $K$  equally spaced test points, the test statistic is computed using the formula

$$C_{RT} = \sum_{k=1}^K \sum_{i=1}^{G-1} \sum_{j=i+1}^G \left| \hat{y}_{ki} - \hat{y}_{kj} \right|$$

---

## Data Structure

The data are entered in two variables: one dependent variable and one independent variable. Additionally, you may specify a frequency variable containing the observation count for each row and a group variable that is used to partition the data into independent groups.

---

## Missing Values

Rows with missing values in the variables being analyzed are ignored in the calculations. When only the value of the dependent variable is missing, predicted values are generated.

## Example 1 – Curve Fitting

This section presents an example of how to fit and compare a Michaelis-Menten model (model 8) to two groups of data. This example will use the data in the **FnReg5** dataset. In this example, the dependent variable is Response, and the independent variable is Temp. The groups are defined by the values of Type.

### Setup

To run this example, complete the following steps:

#### 1 Open the FnReg5 example dataset

- From the File menu of the NCSS Data window, select **Open Example Data**.
- Select **FnReg5** and click **OK**.

#### 2 Specify the Curve Fitting – General procedure options

- Find and open the **Curve Fitting – General** procedure using the menus or the Procedure Navigator.
- The settings for this example are listed below and are stored in the **Example 1** settings template. To load this template, click **Open Example Template** in the Help Center or File menu.

<u>Option</u>	<u>Value</u>
<b>Variables Tab</b>	
Y Variable .....	<b>Response</b>
X Variable .....	<b>Temp</b>
Group Variable.....	<b>Type</b>
Preset Model.....	<b>8 Y=AX/(B+X) Michaelis-Menten</b>
Bootstrap Confidence Intervals .....	<b>Checked</b>
Randomization Hypothesis Tests .....	<b>Checked</b>
<b>Reports Tab</b>	
Combined Summary Report .....	<b>Checked</b>
Group Difference F-Test.....	<b>Checked</b>
Iteration Summary Report .....	<b>Checked</b>
Iteration Detail Report.....	<b>Unchecked</b>
Model Estimation Report .....	<b>Checked</b>
Analysis of Variance Report .....	<b>Checked</b>
Parameter Correlations Report.....	<b>Checked</b>
Predicted Values Report.....	<b>Checked</b>
Residual Report.....	<b>Checked</b>
Predict Y at these X Values.....	<b>5 10 15 20</b>
<b>Resampling Tab</b>	
Samples (N).....	<b>3000</b>
Monte Carlo Samples .....	<b>1000</b>
Random Seed.....	<b>17448</b> (for reproducibility)

#### 3 Run the procedure

- Click the **Run** button to perform the calculations and generate the output.

## Curve Fitting – General

## Parameter Estimates for All Groups

### Parameter Estimates for All Groups

Type	Count	Iter's	R2	A	B
1	21	4	0.98356	10.72798	4.95941
3	21	6	0.97645	10.31200	1.42325
Combined	42	4	0.81153	10.20315	2.54358

This report displays a summary of the results for each group and then for the case in which all groups are combined into one group.

### Group Name (Type)

This column, headed by the name of the Group Variable, lists the group value that is displayed on this line. Note that the Value Labels option may be used to give more meaningful names to these values.

### Count

This is the number of observations used by the nonlinear regression algorithm.

### Iter's

This is the number of iterations used by the nonlinear regression algorithm to find the estimates. You should note whether the maximum number of iterations has been reached (in which case the algorithm did not converge).

### R2

This is the value of the pseudo R-squared value. A value near one indicates that the model fits the data well. A value near zero indicates that the model does not fit the data well.

### A B

The final values of the estimated parameters are displayed so that you may compare them across groups.

## Analysis of Variance Across Groups

### Analysis of Variance Across Groups

Type	Count	Iter's	Model R2	Error DF	Sum Squares Error	Mean Square Error
1	21	4	0.98356	19	1.73157	0.09114
3	21	6	0.97645	19	2.16427	0.11391
Combined	42	4	0.81153	40	43.74009	1.09350
Ignored			0.98321	38	3.89585	0.10252

This report displays goodness of fit results for each group and then for the case in which all groups are combined into one dataset. The final row of the report, labeled 'Ignored', gives the goodness of fit statistics for the model in which a separate curve is fit for each group.

### Group Name (Type)

This column, headed by the name of the Group Variable, lists the group value that is displayed on this line.

### Count

This is the number of observations used by the nonlinear regression algorithm.

### Iter's

This is the number of iterations used by the nonlinear regression algorithm to find the estimates. You should note whether the maximum number of iterations has been reached (in which case the algorithm did not converge).

## Curve Fitting – General

### R2

This is the value of the pseudo R-squared value. A value near one indicates that the model fits the data well. A value near zero indicates that the model does not fit the data well. Note

### Error DF

The degrees of freedom are the number of observations minus the number of parameters fit.

### Sum Squares Error

This is the sum of the squared residuals for this group.

### Mean Square Error

This is a rough estimate of the variance of the residuals for this group.

---

## Curve Inequality F-Test

### Curve Inequality F-Test

Curves Tested	DF	Mean Square	F Ratio	F-Test Prob Level
All	2	19.92212	194.3200	0.00000
Error	38	0.10252		

This report displays an F-Test of whether all of the group curves are equal. This test compares the residual sum of squares obtained when the grouping is ignored with the total of the residual sum of squares obtained for each group. This test is routinely used in analysis linear models and its application to nonlinear models has occasionally been suggested. However, it is based on normality assumptions which seldom occur. When testing curve coincidence is important, we suggest you use a randomization test.

### Curves Tested

This column indicates the term presented on this row.

### DF

The degrees of freedom of this term.

### Mean Square

The mean square associated with this term.

### F Ratio

The F-ratio for testing the hypothesis that all curves coincide.

### F-Test Prob Level

This is the probability level of the F-ratio. When this value is less than 0.05 (a common value for alpha), the test is 'significant' meaning that the hypothesis of equal curves is rejected. If this value is larger than the nominal level (0.05), the null hypothesis cannot be rejected. We do not have enough evidence to reject.

## Curve Inequality Randomization Tests

### Curve Inequality Randomization Tests

Curves Tested	Randomization Prob Level	Monte Carlo Samples	Number of Points Compared Along Curve
1 vs. 3	0.00000	1000	10

User-Entered Random Seed: 17448

This report displays the results of a randomization test whose null hypothesis is that all the group curves coincide. When more than two groups are present, a separate test is provided for each pair of groups, plus a combined test of the equality of all groups.

#### Curves Tested

This column indicates the groups whose equality is being tested on this row.

#### Randomization Prob Level

This is the two-sided probability level of the randomization test. When this value is less than 0.05, the test is 'significant' meaning that the null hypothesis of equal curves is rejected. If this value is larger than the nominal level (0.05), there is not enough evidence in the data to reject the null hypothesis of equality.

(Note: because this is a Monte Carlo test, your results may vary from those displayed here.)

#### Monte Carlo Samples

The number of Monte Carlo samples.

#### Number of Points Compared Along the Curve

The number of values along the X axis at which a comparison between curves is made. Of course, the more X values used, the more accurate (and time consuming) will be the test.

## Parameter Inequality Randomization Tests

### Parameter Inequality Randomization Tests

Curves Compared	Parameter Tested	Randomization Prob Level	Monte Carlo Iterations
1 vs. 3	A	0.70700	1000
1 vs. 3	B	0.00900	1000

User-Entered Random Seed: 17448

This report displays the results of randomization tests about the equality of each parameter across groups. When more than two groups are present, a separate test is provided for each pair of groups, plus a combined test of parameter equality of all groups.

#### Curves Compared

This column indicates the groups being tested on this row.

#### Parameter Test

This column indicates model parameter whose equality is being tested.

## Curve Fitting – General

### Randomization Prob Level

This is the two-sided probability level of the randomization test. When this value is less than 0.05, the test is 'significant' meaning that the null hypothesis of equal parameter values across groups is rejected. If this value is larger than the nominal level (0.05), there is not enough evidence in the data to reject the null hypothesis of equality.

(Note: because this is a Monte Carlo test, your results may vary from those displayed here.)

### Monte Carlo Samples

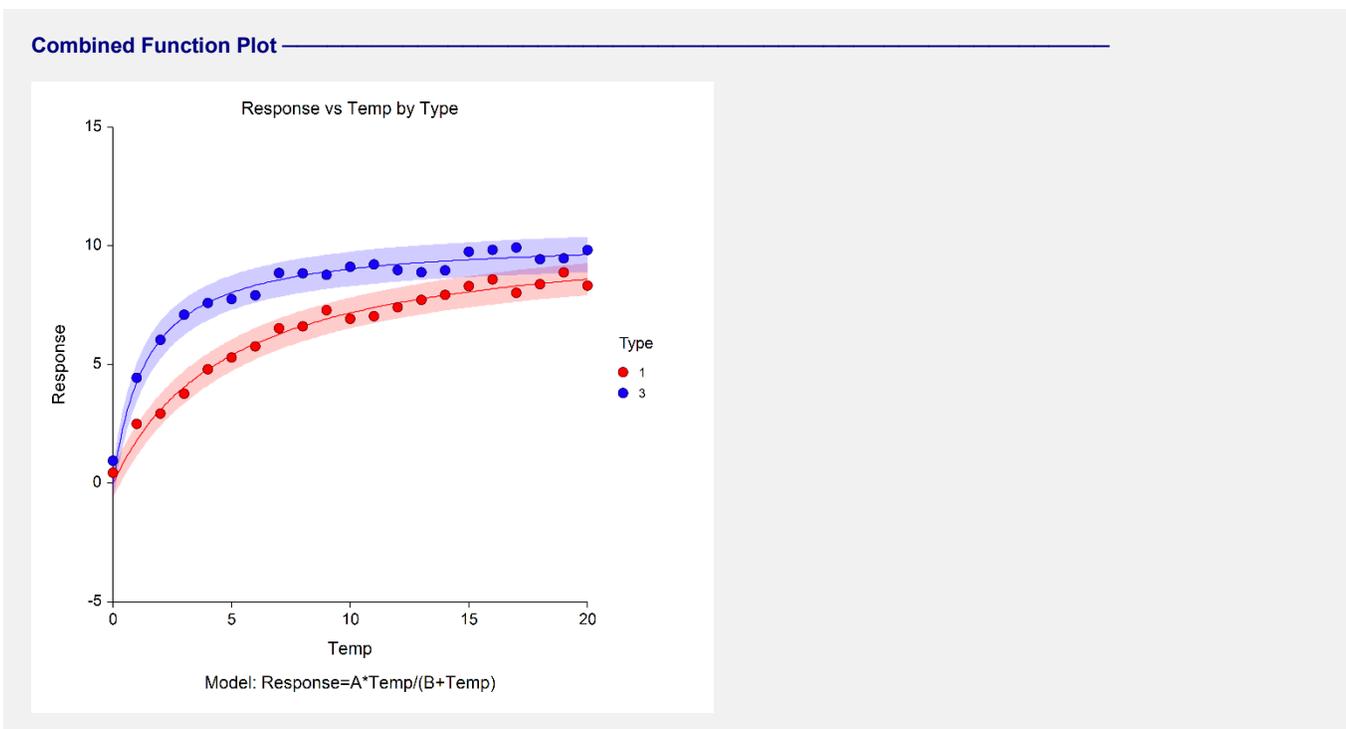
The number of Monte Carlo samples.

### Number of Points Compared Along the Curve

The number of values along the X axis at which a comparison between curves is made. Of course, the more X values used, the more accurate (and time consuming) will be the test.

---

## Combined Function Plot



This plot displays all of the data and fitted curves, allowing you to quickly assess the quality of the results.

## Iteration Summary Section for Type=1

### Iteration Summary Section for Type = 1

Itn No.	Residual Sum of Squares	A	B
1	1.81547	10.51692	4.58046
2	1.73188	10.71254	4.93394
3	1.73157	10.72751	4.95871
4	1.73157	10.72798	4.95941

This report displays the progress of the search algorithm in its search for a solution. It allows you to assess whether the algorithm had indeed converged or whether the program should be re-run with the Maximum Iterations increased or the model changed.

Note that if over ten iterations were needed, the program does not display every iteration.

## Model Estimation Section for Type = 1

### Model Estimation Section for Type = 1

Parameter Name	Parameter Estimate	Asymptotic Standard Error	Lower 95% C.L.	Upper 95% C.L.
A	10.72798	0.30895	10.08135	11.37461
B	4.95941	0.44270	4.03282	5.88599
Iterations	4	Rows Read	21	
R-Squared	0.983564	Rows Used	21	
Random Seed	17448	Total Count	21	

#### Estimated Model

$$(10.7279796046503)^*(x)/((4.95940560324547)+(x))$$

This report displays the details of the estimation of the model parameters.

#### Parameter Name

The name of the parameter whose results are shown on this line.

#### Parameter Estimate

The estimated value of this parameter.

#### Asymptotic Standard Error

An estimate of the standard error of the parameter based on asymptotic (large sample) results.

#### Lower 95% C.L.

The lower value of a 95% confidence limit for this parameter. This is a large sample (at least 25 observations for each parameter) confidence limit. In most cases, the bootstrap confidence interval will be more accurate.

#### Upper 95% C.L.

The upper value of a 95% confidence limit for this parameter. This is a large sample (at least 25 observations for each parameter) confidence limit. In most cases, the bootstrap confidence interval will be more accurate.

#### Iterations

The number of iterations that were completed before the nonlinear algorithm terminated. If the number of iterations is equal to the Maximum Iterations that you set, the algorithm did not converge, but was aborted.

## Curve Fitting – General

**R-Squared**

There is no direct R-squared defined for nonlinear regression. This is a pseudo R-squared constructed to approximate the usual R-squared value used in multiple regression. We use the following generalization of the usual R-squared formula:

$$R\text{-Squared} = (ModelSS - MeanSS)/(TotalSS - MeanSS)$$

where *MeanSS* is the sum of squares due to the mean, *ModelSS* is the sum of squares due to the model, and *TotalSS* is the total (uncorrected) sum of squares of Y (the dependent variable).

This version of R-squared tells you how well the model performs after removing the influence of the mean of Y. Since many nonlinear models do not explicitly include a parameter for the mean of Y, this R-squared may be negative (in which case we set it to zero) or difficult to interpret. However, if you think of it as a direct extension of the R-squared that you use in multiple regression, it will serve well for comparative purposes.

**Random Seed**

This is the value of the random seed that was used when running the bootstrap confidence intervals and randomization tests. If you want to duplicate your results exactly, enter this random seed into the Random Seed box under the Simulation tab.

**Estimated Model**

This is the model that was estimated with the parameters replaced with their estimated values. This expression may be copied and pasted as a variable transformation in the spreadsheet. This will allow you to predict for additional values of X. Note that to ensure accuracy, the parameter estimates are always given to double-precision accuracy.

---

**Analysis of Variance Table for Type = 1**

Analysis of Variance Table for Type = 1			
Source	DF	Sum of Squares	Mean Square
Mean	1	847.88494	847.88494
Model	2	951.50296	475.75148
Model (Adjusted)	1	103.61802	103.61802
Error	19	1.73157	0.09114
Total (Adjusted)	20	105.34959	
Total	21	953.23453	

**Source**

The labels of the various sources of variation.

**DF**

The degrees of freedom.

**Sum of Squares**

The sum of squares associated with this term. Note that these sums of squares are based on Y, the dependent variable. Individual terms are defined as follows:

<b>Mean</b>	The sum of squares associated with the mean of Y. This may or may not be a part of the model. It is presented since it is the amount used to adjust the other sums of squares.
<b>Model</b>	The sum of squares associated with the model.
<b>Model (Adjusted)</b>	The model sum of squares minus the mean sum of squares.
<b>Error</b>	The sum of the squared residuals. This is often called the sum of squares error or just "SSE."

## Curve Fitting – General

**Total** The sum of the squared Y values.

**Total (Adjusted)** The sum of the squared Y values minus the mean sum of squares.

**Mean Square**

The sum of squares divided by the degrees of freedom. The Mean Square for Error is an estimate of the underlying variation in the data.

**Bootstrap Section when Type = 1****Bootstrap Section when Type = 1**

Estimation Results		Bootstrap Confidence Limits		
Parameter	Estimate	Conf. Level	Lower	Upper

<b>A</b>				
Original Value	10.72798	0.90000	10.24470	11.29654
Bootstrap Mean	10.70926	0.95000	10.13571	11.38597
Bias (BM - OV)	-0.01872	0.99000	9.98863	11.59956
Bias Corrected	10.74670			
Standard Error	0.31672			

<b>B</b>				
Original Value	4.95941	0.90000	4.28641	5.77543
Bootstrap Mean	4.93387	0.95000	4.17589	5.91629
Bias (BM - OV)	-0.02554	0.99000	3.93843	6.26731
Bias Corrected	4.98494			
Standard Error	0.45076			

**Predicted Mean and Confidence Limits of Response When Temp = 5.00000**

Original Value	5.38585	0.90000	5.18185	5.52592
Bootstrap Mean	5.39454	0.95000	5.13547	5.54953
Bias (BM - OV)	0.00868	0.99000	5.02522	5.59996
Bias Corrected	5.37717			
Standard Error	0.10463			

**Predicted Mean and Confidence Limits of Response When Temp = 10.00000**

Original Value	7.17139	0.90000	7.06591	7.26638
Bootstrap Mean	7.17153	0.95000	7.04476	7.28754
Bias (BM - OV)	0.00013	0.99000	6.99600	7.32455
Bias Corrected	7.17126			
Standard Error	0.06140			

**Predicted Mean and Confidence Limits of Response When Temp = 15.00000**

Original Value	8.06235	0.90000	7.93663	8.20172
Bootstrap Mean	8.05754	0.95000	7.90776	8.23305
Bias (BM - OV)	-0.00481	0.99000	7.85161	8.28766
Bias Corrected	8.06716			
Standard Error	0.08168			

**Predicted Mean and Confidence Limits of Response When Temp = 20.00000**

Original Value	8.59634	0.90000	8.42315	8.79485
Bootstrap Mean	8.58853	0.95000	8.38480	8.83206
Bias (BM - OV)	-0.00781	0.99000	8.32084	8.92169
Bias Corrected	8.60416			
Standard Error	0.11400			

**Predicted Value and Confidence Limits of Response When Temp = 5.00000**

Original Value	5.38585	0.90000	4.80602	5.80097
Bootstrap Mean	5.39197	0.95000	4.70021	5.84132
Bias (BM - OV)	0.00612	0.99000	4.56952	5.90821
Bias Corrected	5.37973			
Standard Error	0.30743			

## Curve Fitting – General

**Predicted Value and Confidence Limits of Response When Temp = 10.00000**

Original Value	7.17139		0.90000	6.65125	7.56513
Bootstrap Mean	7.16897		0.95000	6.49328	7.59801
Bias (BM - OV)	-0.00243		0.99000	6.39317	7.65040
Bias Corrected	7.17382				
Standard Error	0.29582				

**Predicted Value and Confidence Limits of Response When Temp = 15.00000**

Original Value	8.06235		0.90000	7.52735	8.46999
Bootstrap Mean	8.05498		0.95000	7.38547	8.51379
Bias (BM - OV)	-0.00737		0.99000	7.28565	8.59100
Bias Corrected	8.06972				
Standard Error	0.30082				

**Predicted Value and Confidence Limits of Response When Temp = 20.00000**

Original Value	8.59634		0.90000	8.04041	9.02845
Bootstrap Mean	8.58597		0.95000	7.91382	9.08971
Bias (BM - OV)	-0.01038		0.99000	7.78496	9.19796
Bias Corrected	8.60672				
Standard Error	0.31122				

Sampling Method = Observation, Confidence Limit Type = Reflection, Number of Samples = 3000,  
User-Entered Random Seed = 17448.

This report provides bootstrap estimates and confidence intervals for the parameters, predicted means, and predicted values. Note that bootstrap confidence intervals and prediction intervals are provided for each of the  $X$  (Temp) value requested. Details of the bootstrap method were presented earlier in this chapter.

**Original Value**

This is the parameter estimate obtained from the complete sample without bootstrapping.

**Bootstrap Mean**

This is the average of the parameter estimates of the bootstrap samples.

**Bias (BM - OV)**

This is an estimate of the bias in the original estimate. It is computed by subtracting the original value from the bootstrap mean.

**Bias Corrected**

This is an estimated of the parameter that has been corrected for its bias. The correction is made by subtracting the estimated bias from the original parameter estimate.

**Standard Error**

This is the bootstrap method's estimate of the standard error of the parameter estimate. It is simply the standard deviation of the parameter estimate computed from the bootstrap estimates.

**Conf. Level**

This is the confidence coefficient of the bootstrap confidence interval given to the right.

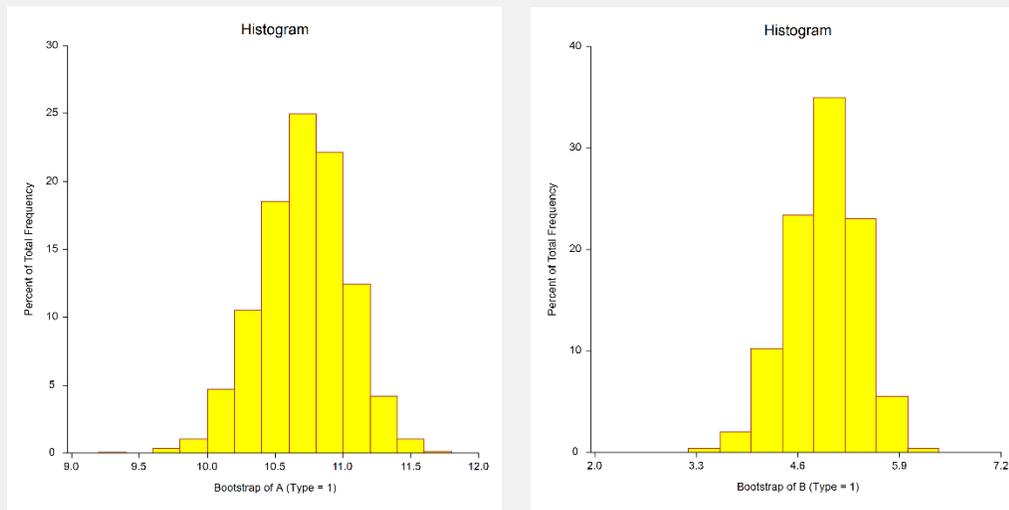
**Bootstrap Confidence Limits - Lower and Upper**

These are the limits of the bootstrap confidence interval with the confidence coefficient given to the left. These limits are computed using the confidence interval method (percentile or reflection) designated on the Bootstrap panel.

Note that to be accurate, these intervals must be based on over a thousand bootstrap samples and the original sample must be representative of the population.

## Bootstrap Histograms for Type = 1

### Bootstrap Histograms for Type = 1



(Several more histograms are displayed.)

Each histogram shows the distribution of the corresponding estimate.

## Asymptotic Correlation Matrix of Parameters

### Asymptotic Correlation Matrix of Parameters for Type = 1

	A	B
A	1.000000	0.940484
B	0.940484	1.000000

This report displays the asymptotic correlations of the parameter estimates. When these correlations are high (absolute value greater than 0.98), the precision of the parameter estimates is suspect.

## Predicted Values for Specified X Values for Type=1

### Predicted Values for Specified X Values for Type = 1

Temp	Predicted Value of Response	Lower 95.0% Prediction Limit	Upper 95.0% Prediction Limit
5.00000	5.38585	4.71548	6.05623
10.00000	7.17139	6.52162	7.82116
15.00000	8.06235	7.40400	8.72069
20.00000	8.59634	7.91914	9.27355

This section shows the predicted mean values and asymptotic (large sample) prediction intervals for the X values that were specified. Note that these are prediction limits for a new value, not confidence limits for the mean of the values.

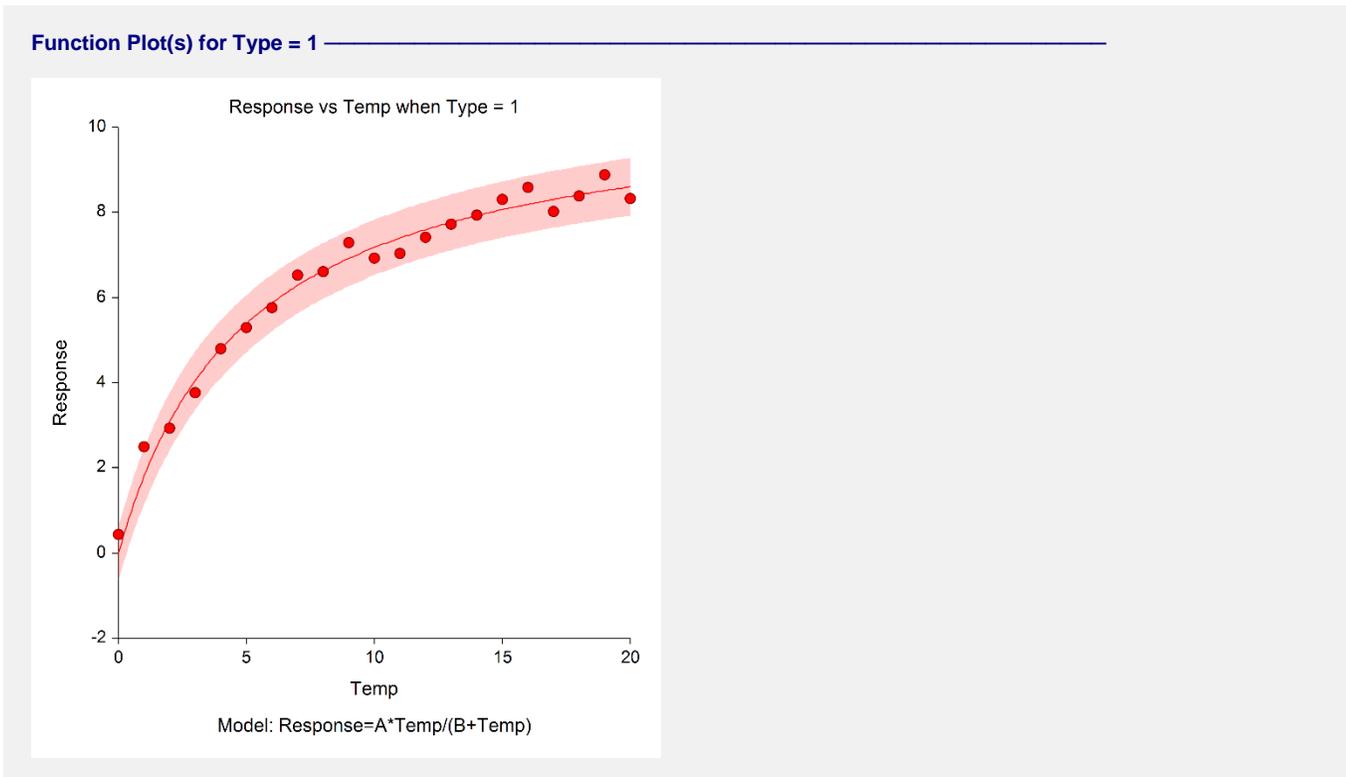
## Predicted Values and Residuals Section

Predicted Values and Residuals Section for Type = 1

Row No.	Temp	Response	Predicted Value	Lower 95.0% Prediction Limit	Upper 95.0% Prediction Limit	Residual
1	0.00000	0.43846	0.00000	-0.63186	0.63186	0.43846
2	1.00000	2.49732	1.80018	1.14295	2.45740	0.69714
3	2.00000	2.93207	3.08302	2.40603	3.76000	-0.15094
4	3.00000	3.76707	4.04351	3.36238	4.72464	-0.27644
5	4.00000	4.79763	4.78959	4.11244	5.46675	0.00803
6	5.00000	5.29474	5.38585	4.71548	6.05623	-0.09111
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.

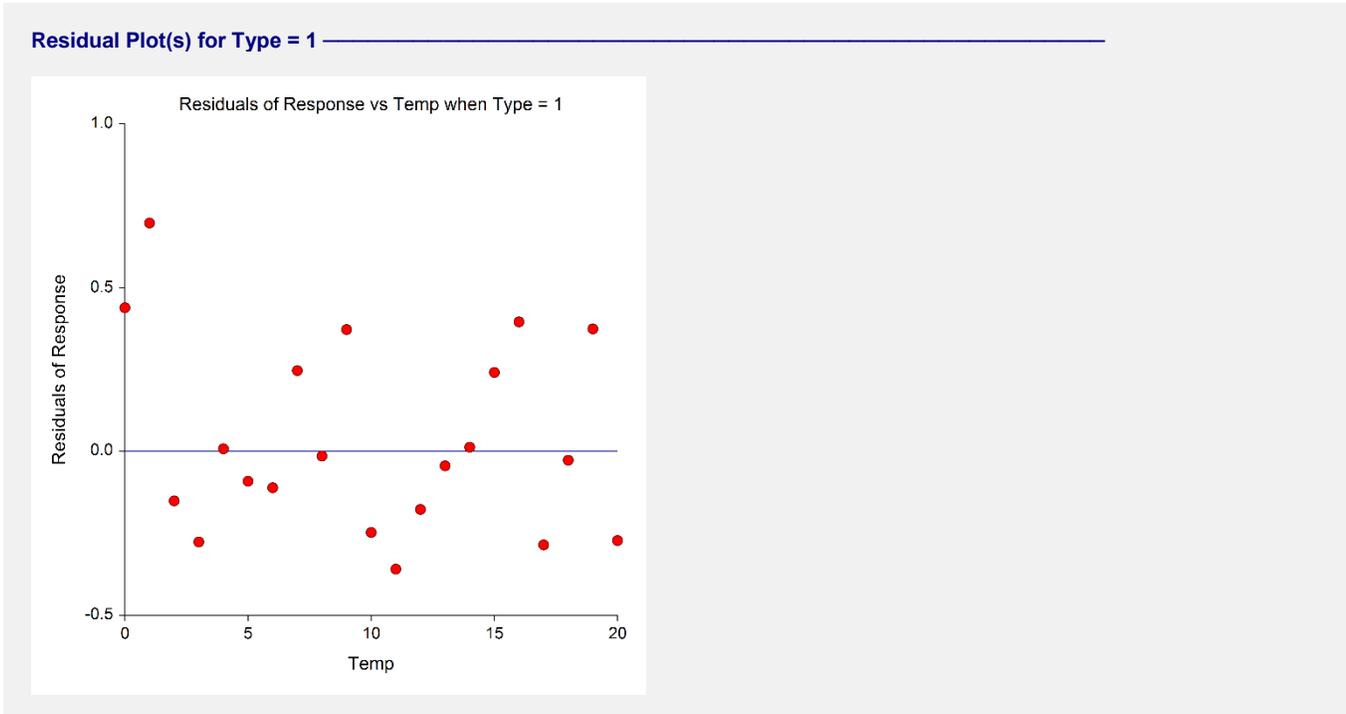
This section shows the values of the predicted values, prediction limits, and residuals. If you have observations in which the independent variable is given, but the dependent (Y) variable is blank, a predicted value and prediction limits will be generated and displayed in this report.

## Function Plot(s) for Type = 1



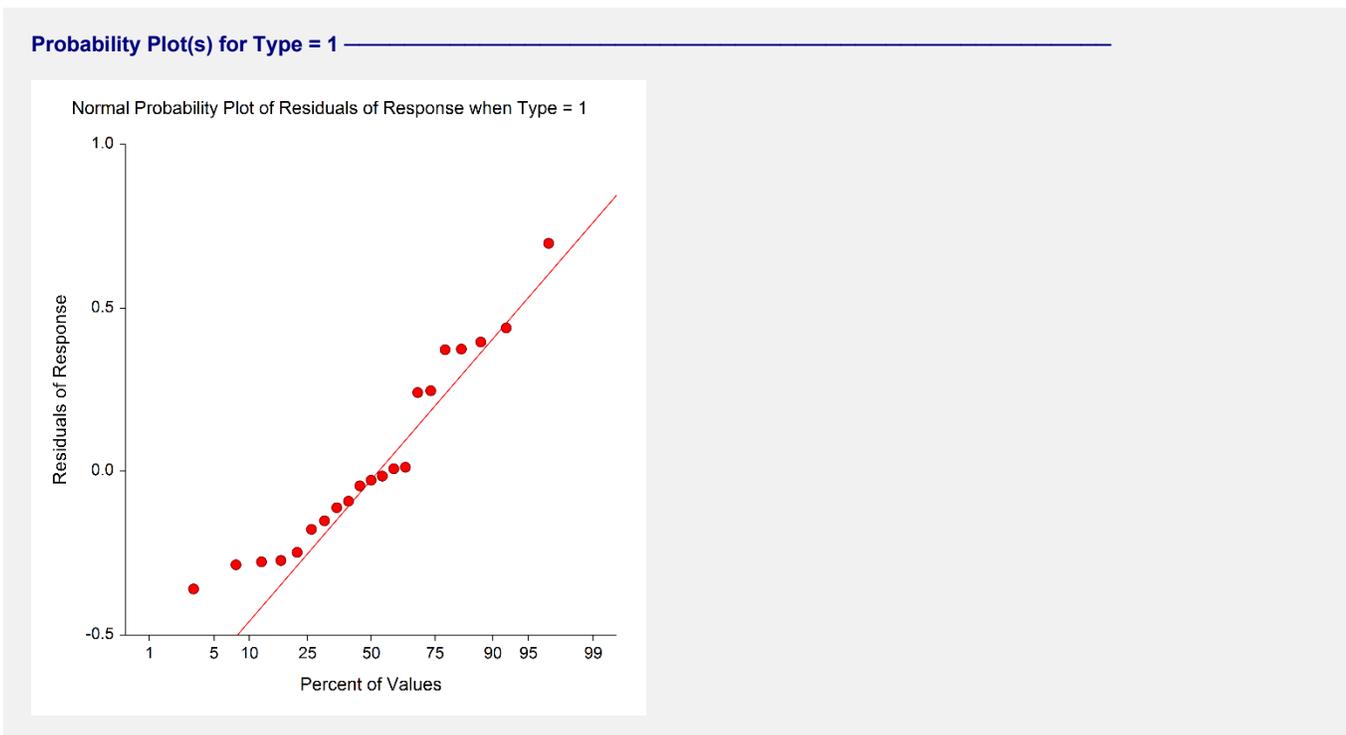
This plot displays the data along with the estimated function. It is useful in deciding if the fit is adequate and the prediction limits are appropriate.

## Residual Plot(s) for Type = 1



This is a scatter plot of the residuals versus the independent variable, X. The preferred pattern is a rectangular shape or point cloud. Any nonrandom pattern may require a redefinition of the model.

## Probability Plot(s) for Type = 1



If the residuals are normally distributed, the data points of the normal probability plot will fall along a straight line. Major deviations from this ideal picture reflect departures from normality. Stragglers at either end of the

## Curve Fitting – General

normal probability plot indicate outliers, curvature at both ends of the plot indicates long or short distributional tails, convex or concave curvature indicates a lack of symmetry, and gaps or plateaus or segmentation in the normal probability plot may require a closer examination of the data or model. We do not recommend that you use this diagnostic with small sample sizes.